

Reconnection of vortex filaments and Kolmogorov spectrum

Sergey K. Nemirovskii

*Institute of Thermophysics, Lavrentyev ave, 1, 630090,
Novosibirsk, Russia and Novosibirsk State University, Novosibirsk*

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The energy spectrum of the 3D velocity field, induced by collapsing vortex filaments is studied. One of the aims of this work is to clarify the appearance of the Kolmogorov type energy spectrum $E(k) \propto k^{-5/3}$, observed in many numerical works on discrete vortex tubes (quantized vortex filaments in quantum fluids). Usually, explaining classical turbulent properties of quantum turbulence, the model of vortex bundles, is used. This model is necessary to mimic the vortex stretching, which is responsible for the energy transfer in classical turbulence. In our consideration we do not appeal to the possible "bundle arrangement" but explore alternative idea that the turbulent spectra appear from singular solution, which describe the collapsing line at moments of reconnection. One more aim is related to an important and intensively discussed topic - a role of hydrodynamic collapse in the formation of turbulent spectra. We demonstrated that the specific vortex filament configuration generated the spectrum $E(k)$ close to the Kolmogorov dependence and discussed the reason for this as well as the reason for deviation. We also discuss the obtained results from point of view of the both classical and quantum turbulence.

Background.—We discuss the possibility of realization of the Kolmogorov type energy spectrum $E(\mathbf{k}) \propto k^{-5/3}$ of the 3D velocity field, produced by the vortex filament, collapsing towards reconnection. The first motivation of this work is related to the problem of modeling classical turbulence with a set of chaotic vortex filaments. This idea has been discussed for quite a long time. (for details see, e.g. [1]-[3]). In classical fluids thin vortex tubes do not exist because they spread due to viscosity, so the concept of vortex filaments should just be considered as a model. Quantum fluids, where the vortex filaments are real objects, give an excellent opportunity for developing the study of the question of whether the dynamics of a set of vortex lines is able to reproduce (at least partially) the properties of real hydrodynamic turbulence. Among various arguments supporting the idea of quasi-classic behaviour of quantum turbulence, the strongest one is the k -dependence of the spectra of energy $E(k)$ obtained in numerical simulations and experiments. There are many works, which demonstrate dependence of $E(k)$ close to the Kolmogorov law $E(k) \propto k^{-5/3}$. These are works, based on the both vortex filament method [4-6], and works using the Gross-Pitaevskii equation [7]-[10]. The most common view of quasi-classical turbulence is the model of vortex bundles. The point is that the quantized vortices have the fixed core radius, so they do not possess the very important property of classical turbulence – stretching of vortex tubes with decrease of the core size. The latter is responsible for the turbulence energy cascade from the large scales to the small scales. Collections of near-parallel quantized vortices (vortex bundles) do possess this property, so the idea that the quasi-classical turbulence in quantum fluids is realized via vortex bundles of different sizes and intensities (number of threads) seems quite natural. Meanwhile, a conception of the bundle structure is vague and up to now it has not been definitely confirmed. It is unclear

how the bundles can spontaneously appear (at low temperature, when the coupling with normal component is small). Moreover, even if they are prepared artificially, they are extremely unstable (see, [3],[11]), they easily can be destroyed in result of reconnection either between of the neighboring threads or in collisions with the other bundles, with the forming of the "bridging". Therefore it quite tempting to find another alternative mechanism of appearing of the Kolmogorov type spectrum, and we offer the collapsing vortex filaments as a candidate for this purpose.

The second motivation is related to other important and intensively discussed topic - a role of hydrodynamic collapse in formation of turbulent spectra. (see e.g. [12], [13]). The striking examples of such type spectra are the Phillips spectrum for water-wind waves, created by white caps – wedges of water surface of water surface or the Kadomtsev-Petviashvili spectrum for acoustic turbulence created by shocks [12]. In the vortex filament theory the singularity formation in a finite time arises due to approach of interacting vortex filaments. The result of this approach is appearing of very acute kink, and energy of interaction between closely located parts can essentially exceed contributions from a smooth elements of lines.

In the work we introduce the general method for calculation of the energy spectrum via the vortex line configuration, then we choose analytic relation for the shape of kink, and conduct the mixed analytic and numerical evaluation of $E(\mathbf{k})$. We demonstrated that the spectrum $E(k)$ is very close to the Kolmogorov dependence $\propto k^{-5/3}$, and discuss the reason of this as well as the reason of deviation.

Calculation of spectrum.—The formal relation, allowing the calculation of $E(\mathbf{k}) = \rho_s \mathbf{v}_\mathbf{k} \mathbf{v}_{-\mathbf{k}} / 2 = \rho_s \boldsymbol{\omega}_\mathbf{k} \boldsymbol{\omega}_{-\mathbf{k}} / 2k^2$ ($\mathbf{k} \mathbf{v}_\mathbf{k} = 0$ due to incompressibility) via the vortex line configuration $\{\mathbf{s}(\xi)\}$, can be written as follows (see

[14],[15])

$$E(\mathbf{k}) = \frac{\rho_s \kappa^2}{16\pi^3 k^2} \oint \oint \mathbf{s}'(\xi_1) \mathbf{s}'(\xi_2) d\xi_1 d\xi_2 e^{i\mathbf{k}(\mathbf{s}(\xi_1) - \mathbf{s}(\xi_2))}. \quad (1)$$

Here $\mathbf{s}(\xi) = \bigcup \mathbf{s}_i(\xi_i)$ is unification of lines $\mathbf{s}_i(\xi_i)$ where $\mathbf{s}_i(\xi_i)$ describes the i -vortex line position parameterized by the label variable ξ_i , $\mathbf{s}'_i(\xi_i)$ denotes the derivative with respect to variable ξ_i (the tangent vector), and $\oint_C = \oint_C \sum_j$. In the isotropic case, the spectral density depends on the absolute value of the wave number k . Integration over the solid angle leads to the formula (see,[15]-[17]):

$$E(k) = \frac{\rho_s \kappa^2}{(2\pi)^2} \oint \oint \mathbf{s}'(\xi_1) \mathbf{s}'(\xi_2) d\xi_1 d\xi_2 \frac{\sin(k |\mathbf{s}(\xi_1) - \mathbf{s}(\xi_2)|)}{k |\mathbf{s}(\xi_1) - \mathbf{s}(\xi_2)|}. \quad (2)$$

For anisotropic situations, formula (2) is understood as the angular average. Thus, for calculation of the energy spectrum $E(k)$ of the 3D velocity field, induced by the collapsing vortex filament we need to know an exact configuration $\{\mathbf{s}(\xi)\}$ of vortex lines.

Shape of kink.—Despite the huge number of works devoted to dynamics of collapsing lines both in classic and quantum fluids [18]-[29] (this list is far not full) the exact solution $\mathbf{s}(\xi)$ for the shape of curves has not obtained up to now. The main results were obtained by different approaches, combining analytical and numerical methods, such as the local induction approximation and full Biot-Savart law, and also Nonlinear Schrödinger equation for vortices in Bose-Einstein condensate. Qualitatively the

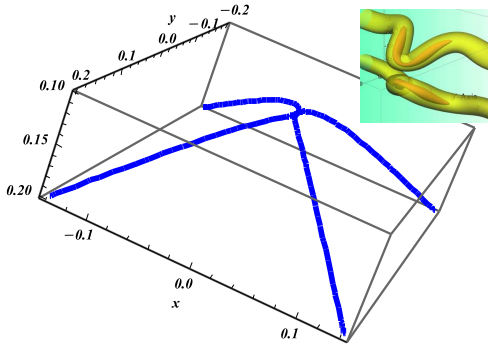


FIG. 1: (Color online) The touching quasi-hyperbolae describing the collapsing lines (see Eq. (3)) obtained in [29]. In the inset we set (as an example) the kinks on the anti-parallel collapsing vortex tubes obtained in numerical simulation [30]

results of these investigations are quite similar and can be described as follows. Due to long range interaction in the Biot-Savart integral, the initially arbitrarily oriented vortices, when they approach each other, they start by re-orienting their close segments so as to bring them into an antiparallel position. Further, cusps may appear on the approaching segments of two vortex lines. The curvature of these cusps may be so large that the self-induced velocity of each perturbation overcomes the repulsion from

the adjoining vortex line. Further the cusps grow, approach each other closer increasing their curvature and correspondingly their self induced velocities and this process is repeated faster and faster. It is important that this process grows explosively, since the distance between the two perturbed segments, Δ , decreases according to the relation $\Delta \sim (t^* - t)^{1/2}$, where t^* is some quantity depending on the relevant parameters and initial conditions. Thus, in a finite time the vortex lines collapse. Asymptotic lines are two hyperbolic curves lying on opposite sides of the pyramid (see e.g. [21],[24]). However, in recent study, [29] it was shown that the curves are not exact hyperbolas, but slightly different lines (the authors called these curves as quasi-hyperbolae) of type $h(\xi) = \sqrt{a^2 \xi^2 / (a^2 + \xi^2) + a^2 + \xi^2}$, and that they lie not in the planes of the pyramid sides, but on the curved surfaces, bent inwards. In the moments just before the collapse, when the vortex cores are nearly touch each other, the very acute kink appears. This curves may be written in parametric form (cf. formula (16) of [29])

$$\mathbf{s}_{1,2}(\xi) = [\pm (h(\xi) - c), \pm \xi, (h(\xi) - b))] \quad (3)$$

The described configuration is shown in Fig. 1. The signs are chosen so that $\mathbf{s}'_1(0) \cdot \mathbf{s}'_2(0) = -1$ (the vortices are antiparallel). Quantity a is of the order of the curvature radius on the tip of the kink of curve, quantity b (related to a , see [29]) is responsible for bending of the surfaces, on which the quasi-hyperbolae lie. Quantity c is also of the order of a is responsible for closeness of the filaments. All this three quantities are smaller of intervortex space $\delta = \mathcal{L}^{-1/2}$ (where \mathcal{L} is the vortex line density). This vision is consistent with the results of numerous numerical works, studying the collapse of vortex lines (see, e.g., [13],[30] and references therein, the decisive picture obtained in [30] is shown in the inset of Fig. 1).

Numerical results.—In the left graphic of Fig. 2 we presented the results of numerical calculation of spectrum $E(k)$ on the base formula (2) (without prefactor before integral) using a configuration $\{\mathbf{s}(\xi)\}$ of vortex lines, described by (3). We chose the following parameters: $a = 0.1, b = 0.09, c = 0.1$ (the case $a \approx c$ corresponds to nearly touching curves). It is seen that in interval of wave numbers k between $1 \div 50$ the slope of $E(k)$ is indeed close to $-5/3$. We discuss the origin of this in the following paragraph.

Analytic consideration.—Because of rapidly oscillating function, the evaluation of integral (2) is difficult, even numerically. In addition, numerical results obscure underlying physics, therefore we intend to perform analytical study, at least as far as possible. The integral (2) can be approximately evaluated for large k using the method of asymptotic expansion [31]. When k is large the function $\sin(k |\mathbf{s}(\xi_1) - \mathbf{s}(\xi_2)|)$ is rapidly varying function, therefore the main contribution into integral comes from points of minimal value of the separation function between points of the curves $D(\xi_1, \xi_2) = |\mathbf{s}(\xi_1) - \mathbf{s}(\xi_2)|$.

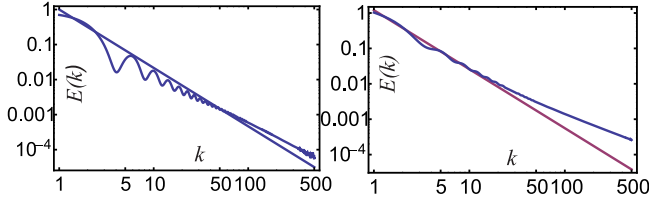


FIG. 2: (Color online) Right. The spectrum $E(k)$, obtained numerically on the base formula (2). The straight line has a slope $-5/3$. Left. The same spectrum obtained on the basis of procedure described in the paragraph "Analytic consideration" (Eq. (4)).

This is enhanced by the fact that the distance is included in the denominator in the integrand of (2). Thus, the behaviour of the phase function $D(\xi_1, \xi_2)$ near minimum is crucial for value of integral and for its k -dependence. Let us study the phase function $D(\xi_1, \xi_2)$ for the vortex configuration described by Eq. (3) just before collapse when $c \approx a$. It is convenient to introduce variables $\rho = \xi_1 - \xi_2$ and $R = (\xi_1 + \xi_2)/2$ and recast the double integral $\int_C \int_C d\xi_1 d\xi_2$ as multiple integral $\int dR \int d\rho$ in the domain bounded by lines $\rho = 2R$ and $\rho = -2R$. The upper limit for R is not essential, since the integral gains the main contribution from vicinity of point $R = 0$. Let's consider the behaviour of function $D(\rho, R)$. It is depicted in Fig 3 (in the extended domain) At the beginning of

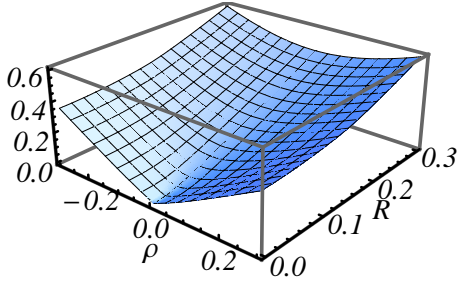


FIG. 3: (Color online) Quantity $D(\rho, R)$ - the phase function in coordinates ρ, R .

coordinates $\rho = 0, R = 0$ function $D(\rho, R) = 0$. The important feature of function $D(\rho, R)$ is its behaviour of it near points $\rho = 0$, (for different R), that is median part of domain, arising from equidistant ($\xi_1 = \xi_2$) points of the touching vortex filaments. For fixed R (perpendicular to the median direction) the functions $D(\rho, R = \text{const})$ are approximated by pieces of parabolas $\propto \rho^2$, then transferring into linear function $\propto |\rho|$ for $\rho \gtrsim a$, with the same slope for all R . Thus, all points of median are points of local minimum, and $\partial D / \partial \rho|_{\rho=0} = 0$ for all R . Another important feature of the phase function $D(\rho, R)$ is its dependence on R along the median $\rho = 0$. Function $D(0, R)$ is depicted in logarithmic coordinated in Fig. 4. It is seen behaves as $\propto R^2$ then transferring into $\propto R^1$ for $R \sim a$ (crossover region). Thus, we have complicated case, when

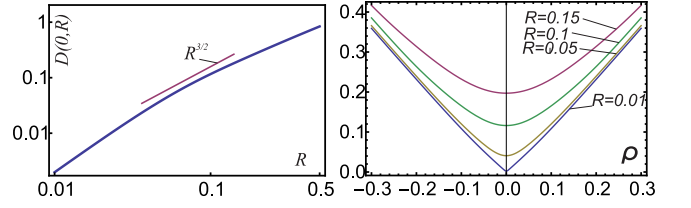


FIG. 4: (Color online) Left. Function $D(0, R)$, distance along the median in the log-log coordinates. Segment of straight line has slope $3/2$. Right. The slices of the phase function $D(\rho, R)$ for different R .

point ($R = 0, \rho = 0$) is simultaneously both a corner of domain restricted by curves $\rho = 2R$ and $\rho = -2R$, and a stationary point (minimum), i.e. $\nabla D(\rho, R) = 0$.

To move further we pass to polar coordinates R, θ , then and integrating over angle θ we obtain asymptotic expansion over $1/k$. The leading term has the form

$$E(k) = \int_0 dR \sqrt{\frac{\pi}{k \partial^2 D / \partial \rho^2|_{\rho=0}}} \frac{\sin(kD(\rho, R))}{kD(\rho, R)} \Big|_{\rho=0} \quad (4)$$

We used hear that integration over θ is alike integration over $d\rho$, namely $d\rho = R d\theta$ and the median curve $\rho = 0$ is the line where function $D(\rho, R = \text{const})$ has a local minimum $\partial D / \partial \rho|_{\rho=0} = 0$. Therefore the integration over θ can be carry out by the use of method of stationary phase, which gives 4. Calculating the integral in vicinity of stationary point we neglected the slowly changing function $\mathbf{s}'(\xi_1) \cdot \mathbf{s}'(\xi_2)$, putting it to be equal to -1 (we recall, that the lines are antiparallel). Additionally, we take $\sin(kD(\rho, R))$ as a imaginary part of $\exp(ikD(\rho, R))$. Thus, we reduced the whole problem to evaluation of the 1D integral. In the right graphic of Fig 2 we presented $E(k)$, calculated on the basis of formula (4). First of all please note that spectrum calculated with use of (4) very close to the spectrum calculated on basis (2), this justifies the approximated procedure, described above. Second, and more important fact is that again in interval of wave numbers k between $1 \div 50$ the slope of $E(k)$ is close to $-5/3$.

To understand an appearance of the $\approx k^{-5/3}$ dependence we appeal to the so called Erdelyi lemma [32], which says that the integral $\int_0 x^{\beta-1} f(x) e^{i\lambda x^\alpha} dx$ with a smooth enough function $f(x)$ has an expansion in asymptotic series as $\sum_m a_m \lambda^{-\frac{m+\beta}{\alpha}}$ with the leading term $\lambda^{-\frac{\beta}{\alpha}}$. That, in particular, means that if we took the collapsing filaments not to be quasi-hyperbolas but was pure power-like functions $\mathbf{s}_{1,2}(\xi) = (\xi, \pm \xi^{3/2}, 0)$ ($3/2$ parabolas) and implemented the procedure described above, we would obtain the spectrum had exact $E(k) \propto k^{-5/3}$ form. Coming back to solution (3) and Fig. (4) we see that intervortex distance (along the median $\xi_1 = \xi_2$) is not $3/2$ parabola but it is more sophisticated function which behaves as $\propto R^2$ then transferring into $\propto R^1$ in the crossover region

Δ , covering $1 \div 1.5$ decades near quantity a . Therefore in the crossover region where the quantity $D(0, R)$ is close to $R^{3/2}$, it should be expected, that $E(k)$ is close to the Kolmogorov dependence $\propto k^{-5/3}$ for the wave numbers k of the order $2\pi/\Delta$, which, indeed, takes a place. The crossover region lies from the scale of bend a , and scale where branches of hyperbolas tends to straight lines. Actually it is close to size of the bridging kink on the curves and is of the order of intervortex space $\delta = \mathcal{L}^{-1/2}$ (see the right picture on Fig. 1). In fact, the numerical works [4]-[10] cited in the Introduction the authors obtained the spectrum $E(k) \approx k^{-5/3}$ only for the wave numbers k , around $k \approx 2\pi/\delta$.

Conclusion.— Coming back to the aims of work stated in the Introduction we can suggest that the spectrum $E(k)$ close to the Kolmogorov dependence $\propto k^{-5/3}$, which was observed in many numerical simulations on the dynamics of quantized vortex filaments [4]-[10], can appear from the reconnecting lines. Unfortunately, because of lack of exact analytic solution for the configuration $\{\mathbf{s}(\xi)\}$ of the collapsing vortex filaments, the quantity $E(k)$ is approximate and relies on the asymptotic solution (3). On the other hand (as seen from the proposed analytical consideration) spectrum depends on few features of collapsing line, such as as order of touching and the crossover to smooth straight line. These features are universal and observed in in many numerical simulations.

Another, more delicate question, touched in the Introduction, concerns the role of dynamics of discrete vortices in the physics of turbulence. On the one hand, our results support the point of view on the role collapse in the formation of turbulent spectra conducted in [12]. On the other hand, many unclear questions remain. In the Kolmogorov scenario the spectrum $E(k) \propto k^{-5/3}$ was the consequence of a k -independent energy cascade P_k in the k space. In the scheme based on collapsing lines the energy cascade does not appear at all (at least in an explicit form). Although due to dimension speculations something like $E(k) \propto (P_k)^{2/3}$ should appear, but the question how it comes, is unclear. It can be put forward an assumption that the collapse of lines, which delivers energy into a tiny region near the point of collapse (then this energy is burned in the process of full reconnection) plays the role of the vortex stretching in the transfer of energy to small scales. But it is just qualitative guess that is not supported by any quantitative calculations. Another question concerns the interval of wave numbers where the spectrum $E(k) \approx k^{-5/3}$ is observed. It is regulated by the curvature of the kink and intervortex space, so in reality this spectrum covers maximum $1 \div 1.5$ decades, while in real turbulence the Kolmogorov dependence $\propto k^{-5/3}$ is observed for $3 \div 4$ decades. Of course, many other issues relating higher structure functions, the number of reconnection, which is necessary to maintain a uniform spectrum, the exact shape of the collapsing curves, etc. remain open, but these issues are outside

the framework of the presented work.

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